P12.9. Joint Production Pricing. Each ton of ore mined from the Baby Doe Mine in Leadville, Colorado, produces one ounce of silver and one pound of lead in a fixed 1:1 ratio. Marginal costs are $10 per ton of ore mined. The demands for silver and lead are

\[ P_S = 11 - 0.00003Q_S \quad \text{and} \quad P_L = 0.4 - 0.000005Q_L \]

where \( Q_S \) is ounces of silver and \( Q_L \) is pounds of lead.

a. Calculate profit-maximizing sales quantities and prices for silver and lead.

First, we have to find total revenue from both goods; (vertical summation)

\[ TR = TR_S + TR_L = (11 - 0.00003Q_S)Q_S + (0.4 - 0.000005Q_L)Q_L. \]

or

\[ TR = 11Q - 0.00003Q^2 + 0.4Q - 0.000005Q^2 \quad \text{(since} \quad Q = Q_S = Q_L). \]

Thus,

\[ MR = 11.4 - 0.00007Q. \]

Now, the profit maximizing output level is found by setting \( MR = MC \):

\[ MR = 11.4 - 0.00007Q = 10 = MC \quad \rightarrow \quad Q = 20000. \]

So, we have \( Q_S = 20000 \) and \( Q_L = 20000 \) which provide positive marginal revenues:

\[ MR_S(Q = 20000) = 11 - 0.00006(20000) = 9.8 \]

and

\[ MR_L(Q = 20000) = 0.4 - 0.00001(20000) = 0.2. \]

And prices for each product are \( P_S = 11 - 0.00003(20000) = 10.4 \) and \( P_L = 0.4 - 0.000005(20000) = 0.3. \)

b. Now assume that wild speculation in the silver market has created a five fold increase in silver demand. Calculate optimal sales quantities and prices for silver and lead.
By the increase in demand, we have to pay five fold price to buy the one ounce of silver, thus, the demand function of silver becomes

\[ P = 5(11 - 0.00003Q_S) = 55 - 0.00015Q_S. \]

Then the procedure is same for optimal quantities and prices as: Since,

\[ MR_T = MR_S + MR_L = 55 - 0.0003Q + 0.4 - 0.00001Q \]

where \( Q = Q_S = Q_L \) and \( MR_T \) is the overall marginal revenue, we can equate \( MR_T = MC = 10 \) to find \( Q \) as

\[ MR = 55.4 - 0.00031Q = 10 \quad \rightarrow \quad Q = 146451.61. \]

Even though \( MR_S(Q_S = 146451.61) = 55 - 0.0003(146451.61) = 11.46 \) is positive, \( MR_L(Q_L = 146451.61) = 0.4 - 0.00001(146451.61) = -1.06 \) is negative. This means that the production and sale of 146451.61 pound of lead decrease revenue, hence, we have dispose some of it. To decide on the optimal amount of sale and disposal, we have to first find the optimal extraction of silver as

\[ MR_S = 55 - 0.0003Q_s = 10 \quad \rightarrow \quad Q^*_S = 150000. \]

By the decision of silver extraction, we have already 150000 pounds of lead but we don’t want to sell all of it (it decreases revenue and of course, the overall profit) but the amount where

\[ MR_L = 0.4 - 0.00001Q_L = 10 \quad \rightarrow \quad Q^*_L = 40000. \]

Thus, we will dump 110000 pounds of lead and sell only 40000 pounds at the price of

\[ P_L = 0.4 - 0.000005(40000) = 0.2. \]

Finally the price of silver is \( P_S = 55 - 0.00015(150000) = 32.5. \)