EXAMPLE: Joint Product Pricing (By-Product): X-Enterprises produces two products in a joint production process. The products are produced in fixed proportions in a 1:1 ratio. The relevant cost functions for this production process is:

\[ TC = 50 + 2Q + 2Q^2 \text{ (and } MC = 2 + 4Q) \]

where \( Q \) is a unit of output consisting of one unit of Product A and one unit of Product B.

Assume the demand curves for two products are

\[ P_A = 352 - Q_A \]
\[ P_B = 100 - 3Q_B \]

Case 1: What are the optimal sales quantities and prices for each of these products? *(Assume unsold production can be costlessly dumped.)*

If each unit of production generates revenues for both products A and B, the appropriate output level is found as:

\[ MR_A + MR_B = MC \]
\[ 352 - 2Q + 100 - 6Q = 2 + 4Q \text{ (Since } Q = Q_A = Q_B) \]
\[ 12Q = 450 \]
\[ Q = 37.5 \]

Thus, profit maximization with equal sales in each market requires that the firm operate at this level \( Q = 37.5 \). Marginal revenues in the two markets are:

\[ MR_A = 352 - 2Q = 352 - 2(37.5) = 277 \]
\[ MR_B = 100 - 6Q = 100 - 6(37.5) = -125 \]

Despite the fact that \( MR_A + MR_B = 277 - 125 = 2 + 4(37.5) = MC \), the negative marginal revenue for B invalidates this solution. In this case,
the company would sell $B$ only up to the point where its marginal revenue is zero since with the production of $A$, the relevant marginal cost of $B$ is zero.

\[
MR_B = MC_B
\]

\[100 - 6Q_B = 0\]

\[6Q_B = 100\]

\[Q_B = 16.67\]

and \[P_B = 100 - 3(16.67) = 49.99\]. Determination of the optimal production and sales level for $A$ is found by equating the marginal revenue from $A$, the only production being sold from the marginal production unit, with the marginal cost of production.

\[
MR_A = MC_A = MC_Q
\]

\[352 - 2Q = 2 + 4Q \text{ (Since } Q = Q_A)\]

\[6Q = 350\]

\[Q = 58.33\]

and \[P_A = 352 - 1(58.33) = 293.67\]. Here, note that \[MR_A + MR_B = 235.33 = MC\], but unlike before \[MR_A = MC_A\] and \[MR_B = MC_B\] as well. Thus the company produces 58.33 units of output, selling 58.33 units of $A$ at price $292.67. Only 16.67 units of $B$ will be sold at price of $49.99, with the remaining 41.67 units being destroyed.

**Case 2:** Assume now that the company incurs an added disposal cost of $20 for the number of units of $A$ and/or $B$ manufactured but not sold. What are the optimal sales quantities and product prices under these conditions?

The solution of Case 1 indicates that only disposal of $B$ needs to be examined. In this situation it will be more profitable for the company to continue selling $B$ so long the negative marginal revenue is less than $20, the per unit disposal cost. Alternatively, the marginal cost of selling as opposed to dumping $B$ is -$20. Thus, we can determine the maximum sales quantity for $B$ under these conditions as:

\[
MR_B = MC_B - \text{Disposal cost saving}
\]

\[100 - 6Q_B = 0 - 20\]

\[6Q_B = 120\]

\[Q_B = 20\]
and \( P_B = $100 - 3 \times 20 = $40 \). The optimal production level is found by selling \( MR_A \) equal to \( MC_Q \) plus the disposal cost of unsold \( B \) being produced and dumped at the margin.

\[
MR_A = MC_Q + \text{Disposal Cost} = MC_A
\]

\[
352 - 2Q = 2 + 4Q + 20 \quad \text{(Since} \quad Q = Q_A) \]

\[
6Q = 330
\]

\[
Q = 55
\]

and \( P_A = 352 - 1(55) = $297 \). Once again, \( MR_A + MR_B = MC \). with \( MR_A = MC_A \) and \( MR_B = MC_B \). The company produces 55 units of output selling all 55 units of \( A \) produced and dumping 35 units of \( B \) produced.